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On the self-induced motion of a helical vortex

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The velocity field in the immediate vicinity of a curved vortex comprises a circulation around the vortex, a component due to the vortex curvature, and a 'remainder' due to the more distant parts of the vortex. The first two components are relatively well understood but the remainder is known only for a few specific vortex geometries, most notably, the vortex ring. In this paper we derive a closed form for the remainder that is valid for all values of the pitch of an infinite helical vortex. The remainder is obtained firstly from Hardin's (1982) solution for the flow induced by a helical line vortex (of zero thickness). We then use Ricca's (1994) implementation of the Moore & Saffman (1972) formulation to obtain the remainder for a helical vortex with a finite circular core over which the circulation is distributed uniformly. It is shown analytically that the two remainders differ by $1/4$ for all values of the pitch. This generalizes the results of Kuibin & Okulov (1998) who obtained the remainders and their difference asymptotically for small and large pitch. An asymptotic analysis of the new closed-form remainders using Mellin transforms provides a complete representation by a residue series and reveals a minor correction to the asymptotic expression of Kuibin & Okulov (1998) for the remainder at small pitch.

1. Introduction

Helical vortices are important for at least three reasons. First, they model the tip vortices behind propellers, wind turbines and rotors in hover or vertical flight. It is usually assumed that, sufficiently far from the blades, the wake is fully developed and the tip vortices can be considered as infinite helical vortices of constant radius and pitch. In that case, the velocity field in the wake depends on the pitch as well as the circulation of the vortices (Wood 1998), and so their geometry and behaviour is relevant to machine performance. In particular for wind turbines, it is important to improve wake modelling in the high-thrust, small-pitch region where the traditional equations that lead, for example, to the Betz limit, break-down.

Secondly, infinite helical vortices of constant pitch p – the only type considered in this paper – represent the next level of geometrical complexity after the circular vortex ring. The vortex ring is the prototype geometry for analysing the combined effects of curvature and core structure, and is, therefore, treated extensively in Saffman's (1992) monograph. Mathematically, curvature gives rise to a logarithmic singularity in the expression for the self-induced motion of a vortex; see e.g. Ricca (1994, formula (1.2)) and Kuibin & Okulov (1998, formula (1)). In comparison to the vortex ring, the helical vortex introduces the effects of torsion and, particularly at small pitch, provides an example of the importance of the remainder term, denoted as Q_f in the expressions just referred to. The velocity of a vortex ring is dominated by the logarithmic term, but Kuibin & Okulov (1998) and Wood (1998) show that, for a helix, $Q_f \sim p^{-1}$ as $p \downarrow 0$.

Thirdly, helical vortices, like vortex rings, are one of a small number of vortex geometries that can translate without deformation. This means that the self-induced velocity of primary interest is in the binormal direction; in particular, the remainder term for the binormal velocity is the subject of this paper. The starting point for the analysis and the necessary background in the geometry of helical vortices is provided by Ricca (1994). For the sake of brevity, we will omit the details that can be found in the sections of his paper to which we refer.

The expression for the self-induced velocity of a helix can be obtained either from Hardin's (1982) solution for the inviscid flow induced by a line helical vortex (of zero thickness), or from the Moore & Saffman (1972) method of directly treating the Biot–Savart law. In the latter method, the singularity is removed by subtracting the effects of the osculating circular vortex, whereupon the known velocity due to this vortex is added; see also Saffman (1992, § 11.4). The primary restriction on the Moore & Saffman procedure is that the diameter of the vortex core remain small compared to its radius of curvature. This is, apparently, the case for helical vortices trailing from wind turbines and propellers, e.g. Wood (1998). The procedure can be viewed as a generalization of the simple 'cut-off' method of removing the curvature singularity in the Biot–Savart law by altering the limits of the integral. Since the added velocity due to the osculating vortex depends on the structure (its core size, distribution of vorticity, etc.) of the vortex, the velocities obtained by the two methods will usually differ. In this paper we determine the remainder terms for the binormal velocity, for which we will use Ricca's (1994) notation of C_H and C_{MS} for the Hardin (1982) and the Moore & Saffman (1972) terms, respectively. In the next section, we derive a closed-form solution for C_H that is valid for all values of p. In § 3 we start from Ricca (1994, formulae (3.15)–(3.17)) for a circular-core vortex over which the vorticity is distributed uniformly, and derive a closed-form solution for C_{MS}. The important result that C_H and C_{MS} differ by $1/4$ for all p follows immediately. Section 4 considers the asymptotics of the new solutions and the numerical treatment of the integral appearing in the solutions; the integral cannot be evaluated in closed form. Specifically, we use Mellin transforms to generate the asymptotic expansions for small and large pitch. The expansion for large pitch is consistent with the well-known result due to Kelvin (1880). Our expression for small pitch provides a minor correction to that of Kuibin & Okulov (1998).

In this paper, all lengths are normalized by the vortex radius (not the radius of the vortex core), so that, for example, the pitch, p, is the normalized axial distance traversed by a material point as the vortex angle increases by 2π. In conformity with Ricca (1994) and Kuibin & Okulov (1998), all velocities are scaled by $\Gamma/4\pi$, where Γ is the circulation of the vortex. The radial direction is r, and ε is the always positive radial distance from the vortex to any point in the flow, located at the same vortex angle.

2. Self-induced velocity obtained from Hardin's (1982) solution

Hardin (1982) obtained the general solution for the inviscid flow induced by a helical line vortex in terms of separate solutions valid for the interior flow ($r<1$) and the exterior flow ($r>1$). In this section we adopt Ricca's (1994, § 4.1.1) re-elaboration of Hardin's solution for the binormal velocity at a distance ε exterior, or interior, to the vortex. The exterior and interior solutions involve a Kapteyn series which is analysed (for $\varepsilon \downarrow 0$) by the method of Boersma & Yakubovich (1998). In this manner we determine the remainder term, C_H, as the common limit of C_{ext} and C_{int} (the exterior and interior forms of the remainder term), when $\varepsilon \downarrow 0$.

According to Ricca (1994, formulae (4.10)–(4.12)) with an appropriate change of notation, C_{ext} is given by

$$C_{ext} = -\frac{2p(1+p^2)^{1/2}}{1+\varepsilon(1+p^2)} - \frac{4(1+\varepsilon)(1+p^2)^{3/2}}{p^2[1+\varepsilon(1+p^2)]}\mathscr{P}(\varepsilon,p) + \frac{2}{\varepsilon} + \log\varepsilon, \tag{2.1}$$

where

$$\mathscr{P}(\varepsilon,p) = \sum_{\nu=1}^{\infty} \nu K_\nu(\nu[1+\varepsilon(1+p^2)]/p)\, I'_\nu(\nu/p). \tag{2.2}$$

Here, $I_\nu(\,\cdot\,)$ and $K_\nu(\,\cdot\,)$ are modified Bessel functions of order ν, and the prime denotes differentiation with respect to the argument. The series (2.2) is a Kapteyn series which we will now analyse for $\varepsilon \downarrow 0$. Introduce the notation

$$S(a,b) = \sum_{m=1}^{\infty} K_m(ma)\, I_m(mb), \tag{2.3}$$

where the series is convergent for $a > b \geqslant 0$. Then the Kapteyn series (2.2) is equal to the derivative $\partial S(a,b)/\partial b$, with $a = [1+\varepsilon(1+p^2)]/p$ and $b = 1/p$. Boersma & Yakubovich (1998) established the integral representation

$$S(a,b) = \frac{1}{2}\int_0^\infty \left[(t^2+a^2+b^2-2ab\cos t)^{-1/2} - \frac{1}{\pi}\int_0^\pi (t^2+a^2+b^2-2ab\cos s)^{-1/2}\,\mathrm{d}s\right]\mathrm{d}t, \tag{2.4}$$

which yields, by differentiation with respect to b,

$$\frac{\partial S(a,b)}{\partial b} = -\frac{1}{2}\int_0^\infty \frac{b-a\cos t}{(t^2+a^2+b^2-2ab\cos t)^{3/2}}\,\mathrm{d}t + \frac{1}{2\pi}\int_0^\infty \mathrm{d}t \int_0^\pi \frac{b-a\cos s}{(t^2+a^2+b^2-2ab\cos s)^{3/2}}\,\mathrm{d}s. \tag{2.5}$$

Here, the second (double) integral turns out to vanish, as can be seen by interchanging the order of integration and using the auxiliary integrals

$$\int_0^\infty \frac{\mathrm{d}t}{(t^2+a^2+b^2-2ab\cos s)^{3/2}} = \frac{1}{a^2+b^2-2ab\cos s},$$

$$\frac{1}{2\pi}\int_0^\pi \frac{b-a\cos s}{a^2+b^2-2ab\cos s}\,\mathrm{d}s = 0, \quad a > b \geqslant 0.$$

Substituting $a = b(1+\varepsilon')$ in (2.5), with $\varepsilon' = \varepsilon(1+p^2)$, we find

$$\left.\frac{\partial S(a,b)}{\partial b}\right|_{a=b(1+\varepsilon')} = -\frac{1}{2b}\int_0^\infty \frac{1-(1+\varepsilon')\cos bt}{[t^2+\varepsilon'^2+2(1+\varepsilon')(1-\cos bt)]^{3/2}}\,\mathrm{d}t, \tag{2.6}$$

where the change in integration variable from t to bt reinforces the similarity with the

Biot–Savart integral. The behaviour of the integral (2.6) as $\varepsilon' \downarrow 0$, was investigated by Boersma & Yakubovich (1998). Their final result, translated to the present notation, reads

$$\left.\frac{\partial S(a,b)}{\partial b}\right|_{a=b(1+\varepsilon')} = \frac{1}{2b(1+b^2)^{1/2}}\frac{1}{\varepsilon'} + \frac{b}{4(1+b^2)^{3/2}}\log\varepsilon'$$
$$-\frac{b}{4(1+b^2)^{3/2}}\log(2(1+b^2)^{1/2})$$
$$-\frac{1}{2b}\int_0^\infty\left\{\frac{1-\cos bt}{[t^2+2(1-\cos bt)]^{3/2}}\right.$$
$$\left.-\frac{b^2}{2(1+b^2)^{3/2}}\frac{H(1-t)}{t}\right\}\mathrm{d}t + o(1), \quad (\varepsilon'\downarrow 0) \tag{2.7}$$

in which $H(\cdot)$ denotes the unit step function, and $o(1)$ is Landau's notation for an expression that tends to 0 when $\varepsilon' \downarrow 0$.

Recall that the derivative $\partial S(a,b)/\partial b$ is related to $\mathscr{P}(\varepsilon,p)$ from (2.2). Thus, by substituting $b = 1/p$, $\varepsilon' = \varepsilon(1+p^2)$, in (2.7), we obtain the small-ε expansion

$$\mathscr{P}(\varepsilon,p) = \frac{p^2}{2(p^2+1)^{3/2}}\frac{1}{\varepsilon} + \frac{p^2}{4(p^2+1)^{3/2}}\log\varepsilon + \frac{p^2}{4(p^2+1)^{3/2}}\log[(p^2+1)^{1/2}/2]$$
$$-\tfrac{1}{4}p^2W(p) + o(1), \quad (\varepsilon\downarrow 0) \tag{2.8}$$

where

$$W(p) = \int_0^\infty\left\{\frac{2(1-\cos t)}{[p^2t^2+2(1-\cos t)]^{3/2}} - \frac{1}{(p^2+1)^{3/2}}\frac{H(1-t)}{t}\right\}\mathrm{d}t. \tag{2.9}$$

Next, we insert (2.8) into (2.1), re-expand for small ε, and identify the $\varepsilon\downarrow 0$-limit of C_{ext} with C_H. As a result we find

$$C_H = \log 2 + 2p^2 - 2p(p^2+1)^{1/2} - \tfrac{1}{2}\log(p^2+1) + (p^2+1)^{3/2}W(p), \tag{2.10}$$

which is our new closed-form expression for C_H, valid for all values of p. In § 4 we examine the asymptotics of $W(p)$ and C_H, both for large and small p. In addition, we consider the numerical evaluation of $W(p)$ which is straightforward except at very small values of p.

To complete this Section, we briefly discuss the $\varepsilon\downarrow 0$-limit of C_{int}, given by (Ricca 1994, formulae (4.8), (4.9), (4.12))

$$C_{int} = \frac{2(1+p^2)^{1/2}}{p} - \frac{4(1-\varepsilon)(1+p^2)^{3/2}}{p^2[1-\varepsilon(1+p^2)]}\mathscr{Q}(\varepsilon,p) - \frac{2}{\varepsilon} + \log\varepsilon, \tag{2.11}$$

where

$$\mathscr{Q}(\varepsilon,p) = \sum_{\nu=1}^{\infty}\nu K_\nu'(\nu/p)I_\nu(\nu[1-\varepsilon(1+p^2)]/p). \tag{2.12}$$

From the analysis of Boersma & Yakubovich (1998) we obtain the following small-ε expansion for the Kapteyn series (2.12):

$$\mathscr{Q}(\varepsilon,p) = -\frac{p^2}{2(p^2+1)^{3/2}}\frac{1}{\varepsilon} + \frac{p^2}{4(p^2+1)^{3/2}}\log\,\varepsilon + \frac{p^2}{4(p^2+1)^{3/2}}\log\,[(p^2+1)^{1/2}/2]$$
$$+\tfrac{1}{2}p - \tfrac{1}{4}p^2W(p) + o(1), \quad (\varepsilon\downarrow 0) \tag{2.13}$$

where $W(p)$ is given by (2.9). Obviously, the $1/\varepsilon$-terms in (2.8) and (2.13) – which are opposite – relate to the circulation around the vortex, whereas the $\log \varepsilon$-terms – which are equal – relate to the curvature. The remaining terms are identical except for the term $p/2$ in (2.13), which arises from the constant difference between Hardin's (1982) exterior and interior solutions; see Hardin (1982, formulae (8), (9) for w). On inserting (2.13) into (2.11) and letting $\varepsilon \downarrow 0$, we get again the limit result (2.10), as anticipated at the outset.

3. Self-induced velocity obtained from the Moore–Saffman procedure

Ricca (1994, § 3) derived the remainder term, C_{MS}, arising from the procedure of Moore & Saffman (1972). The latter authors remove the singularity in the Biot–Savart integral by subtracting the effects of the osculating vortex ring of the same structure, and then adding the known contribution from that ring. Ricca described the formulation and its implementation in detail which we will not reproduce. The starting point of our analysis is Ricca (1994, formulae (3.15)–(3.17)) for an osculating vortex with a circular core over which the vorticity is distributed uniformly. In our notation, these expressions can be combined as

$$C_{MS} = -\tfrac{1}{4} + 3\log 2 + 2\int_0^\infty \left\{ (p^2+1)^{1/2}\, \frac{p^2 t \sin t + (1-p^2)(1-\cos t)}{[p^2t^2 + 2(1-\cos t)]^{3/2}} - \frac{1}{2^{3/2}(p^2+1)^{1/2}}\, \frac{H(\pi(p^2+1)^{1/2} - t)}{[1-\cos(t/(p^2+1)^{1/2})]^{1/2}} \right\} \mathrm{d}t. \tag{3.1}$$

Ricca showed that the expression (3.1) reduces to the result first derived by Kelvin (1880) for large pitch – the Kelvin limit – and then evaluated the integral numerically for smaller pitch. We now show that the integral in (3.1) can be significantly simplified before resorting to numerical analysis.

The second part of the integral in (3.1) can be simplified as follows:

$$\begin{aligned}
&\frac{1}{2^{3/2}(p^2+1)^{1/2}} \int_\delta^{\pi(p^2+1)^{1/2}} \frac{\mathrm{d}t}{[1-\cos(t/(p^2+1)^{1/2})]^{1/2}} \\
&\quad = \frac{1}{2^{3/2}} \int_{\delta/(p^2+1)^{1/2}}^{\pi} \frac{\mathrm{d}t}{(2\sin^2 \frac{1}{2}t)^{1/2}} = -\tfrac{1}{2} \log \tan\left(\frac{\delta}{4(p^2+1)^{1/2}} \right) \\
&\quad = \tfrac{1}{2}\log(4(p^2+1)^{1/2}) - \tfrac{1}{2}\log\delta + O(\delta^2) \\
&\quad = \frac{1}{2}\int_\delta^1 \frac{\mathrm{d}t}{t} + \log 2 + \tfrac{1}{4}\log(p^2+1) + O(\delta^2).
\end{aligned} \tag{3.2}$$

Inserting (3.2) into (3.1) we obtain

$$\begin{aligned}
C_{MS} &= -\tfrac{1}{4} + \log 2 - \tfrac{1}{2}\log(p^2+1) \\
&\quad + \int_0^\infty \left\{ 2(p^2+1)^{1/2}\, \frac{p^2 t\sin t + (1-p^2)(1-\cos t)}{[p^2t^2 + 2(1-\cos t)]^{3/2}} - \frac{H(1-t)}{t} \right\} \mathrm{d}t \\
&= -\tfrac{1}{4} + \log 2 - \tfrac{1}{2}\log(p^2+1) + (p^2+1)^{3/2} W(p) \\
&\quad + 2p^2(p^2+1)^{1/2} \int_0^\infty \frac{t\sin t - 2(1-\cos t)}{[p^2t^2 + 2(1-\cos t)]^{3/2}}\, \mathrm{d}t,
\end{aligned} \tag{3.3}$$

where $W(p)$ is given by (2.9). The final integral in (3.3) is elementary because the integrand has a primitive function, so that

$$\int_0^\infty \frac{t\sin t - 2(1-\cos t)}{[p^2t^2 + 2(1-\cos t)]^{3/2}}\,\mathrm{d}t = \left.\frac{-t}{[p^2t^2 + 2(1-\cos t)]^{1/2}}\right|_0^\infty$$
$$= -\frac{1}{p} + \frac{1}{(p^2+1)^{1/2}}. \tag{3.4}$$

Combining (3.3) and (3.4) we find

$$C_{MS} = -\tfrac{1}{4} + \log 2 + 2p^2 - 2p(p^2+1)^{1/2} - \tfrac{1}{2}\log(p^2+1) + (p^2+1)^{3/2}W(p), \tag{3.5}$$

and a comparison with (2.10) gives immediately

$$C_{MS} = C_H - \tfrac{1}{4} \tag{3.6}$$

for any value of the vortex pitch, p. The results (3.5) and (3.6) form the main contribution of this paper. The identity (3.6) was shown to hold asymptotically for both small and large p by Kuibin & Okulov (1998), and was found by Ricca (1994) to be consistent with his numerical results.

It must be emphasized that the term 1/4 in (3.5) and (3.6) follows from the specific structure of the osculating vortex: a circular core over which the vorticity is distributed uniformly. It can be shown (see e.g. Saffman 1992, § 11.4), that the inclusion of additional effects, such as flow along the axis of the vortex, modifies the term in a straightforward manner. The only restriction on the ability to so modify (3.5) and (3.6) would appear to be contained within the fundamental restriction of the Moore–Saffman procedure, namely, that the core radius, a, is small compared to the helix curvature, that is, $a \ll 1$ for helices of small pitch.

It is believed that the integral $W(p)$ cannot be evaluated in closed form. Therefore, in the next section we consider the asymptotic behaviour of $W(p)$ and C_{MS}, and the numerical evaluation of $W(p)$ over the useful range of pitch values.

4. Asymptotics and numerical evaluation of $W(p)$

The expressions (2.10) for C_H and (3.5) for C_{MS} contain the integral $W(p)$, which cannot be evaluated in closed form. We now examine the behaviour of $W(p)$ for large and small pitch, as did Kuibin & Okulov (1998). They began, in each case, by asymptotically expanding the Bessel functions that appear in (2.2). Both Ricca (1994) and Kuibin & Okulov (1998) show that the large-p form is consistent with the Kelvin limit, which can be derived directly from the Biot–Savart law; see e.g. Saffman (1992, § 11.2). Thus the large-p limit is well known. However, the small-p limit which is much more important and interesting, is less well understood. In this section we establish the asymptotics of $W(p)$ in a combined treatment of the two cases of large and small p. The Mellin-transform method to be employed furnishes the complete asymptotic expansions of $W(p)$ both for large and small p.

We begin by rewriting $W(p)$ from (2.9) as

$$W(p) = \int_0^\infty \left\{ \frac{\sin^2 t}{(p^2t^2 + \sin^2 t)^{3/2}} - \frac{1}{(p^2+1)^{3/2}}\,\frac{H(1/2 - t)}{t} \right\} \mathrm{d}t. \tag{4.1}$$

The asymptotic expansion of $W(p)$ for large p is readily found by expansion of the integrand in (4.1) in powers of p^{-1}, followed by a term-by-term integration. However, the small-p limit is not so easily obtained. As an alternative, we take the Mellin transform of $W(p)$, defined by

$$\mathscr{M}\{W(p)\} = \int_0^\infty W(p)\, p^{z-1}\mathrm{d}p \tag{4.2}$$

for complex z. To evaluate $\mathscr{M}\{W(p)\}$, we need the auxiliary result

$$\mathscr{M}\{(p^2+a^2)^{-3/2}\} = a^{z-3}\,\frac{\Gamma(z/2)\Gamma(3/2-z/2)}{2\,\Gamma(3/2)}, \tag{4.3}$$

valid for $0 < \mathrm{Re}(z) < 3$. Here, $\Gamma(\,\cdot\,)$ denotes the gamma function. Thus we find

$$\mathscr{M}\{W(p)\} = \frac{\Gamma(z/2)\Gamma(3/2-z/2)}{2\,\Gamma(3/2)}\,L(z), \quad 1 < \mathrm{Re}\,(z) < 3, \tag{4.4}$$

where

$$L(z) = \int_0^\infty \left[\frac{|\sin t|^{z-1}}{t^z} - \frac{H(1/2-t)}{t}\right]\mathrm{d}t. \tag{4.5}$$

Obviously, $L(z)$ is analytic for $\mathrm{Re}(z) > 1$ and this explains the range of validity in (4.4). The relevant aspects of the behaviour of $L(z)$ are discussed in the Appendix. By means of the Mellin inversion formula we arrive at the representation

$$W(p) = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(z/2)\Gamma(3/2-z/2)}{2\,\Gamma(3/2)}\,L(z)\,p^{-z}\mathrm{d}z, \quad 1 < c < 3. \tag{4.6}$$

The integrand in (4.6) is analytic to the right of the contour, except for simple poles at $z = 2k+3$, $k = 0, 1, 2, \ldots$, with residues

$$\operatorname*{Res}_{z=2k+3} \frac{\Gamma(z/2)\Gamma(3/2-z/2)}{2\,\Gamma(3/2)}\,L(z)\,p^{-z} = (-1)^{k+1}\frac{(3/2)_k}{k!}\,L(2k+3)\,p^{-2k-3}, \tag{4.7}$$

where Pochhammer's symbol $(3/2)_k$ is defined by

$$(3/2)_0 = 1, \quad (3/2)_k = \tfrac{3}{2}\cdot\tfrac{5}{2}\cdots(k+\tfrac{1}{2}) \quad \text{for } k = 1, 2, 3, \ldots.$$

By closing the contour in (4.6) to the right, we are led to the large-p representation of $W(p)$ by the residue series

$$W(p) \sim \sum_{k=0}^{\infty}(-1)^k\frac{(3/2)_k}{k!}\,L(2k+3)\,p^{-2k-3} \quad (p\to\infty) \tag{4.8}$$

which is identical to the result obtained by the approach indicated below (4.1). It can be shown that the sequence $\{L(2k+3)\}$ is decreasing, with $L(2k+3) \geqslant A - \frac{1}{2}\log(k+1)$ for $k = 0, 1, 2, \ldots$, where A is some constant. Consequently, the asymptotic series (4.8) is convergent for $p > 1$, and $\sim$ may be replaced by an equality sign. With $L(3)$ and $L(5)$ given by (A 17) and (A 18), we obtain the large-p expansion

$$W(p) = (-E+\tfrac{3}{2})p^{-3} - \tfrac{3}{2}(-E+\tfrac{25}{12}-\tfrac{4}{3}\log 2)p^{-5} + O(p^{-7}) \quad (p\to\infty) \tag{4.9}$$

where $E = 0.5772\ldots$ is Euler's constant. Substitution of the first term on the right of (4.9) into the expression (3.5) for C_{MS} gives the Kelvin limit; see e.g. Ricca (1994, formula (3.20)) and Kuibin & Okulov (1998, formula (24)).

Next, we determine the asymptotic expansion of $W(p)$ for small p by closing the contour in (4.6) to the left. This requires knowledge of the singularities of $L(z)$ in the half-plane $\mathrm{Re}(z) \leqslant 1$. The important results are derived in the Appendix where, for example, it is shown that $L(z)$ has simple poles at $z = 1$ and $z = -2k$, $k = 0, 1, 2, \ldots$, with residues

$$\operatorname*{Res}_{z=1} L(z) = 1, \quad \operatorname*{Res}_{z=-2k} L(z) = -1. \tag{4.10}$$

Consider now the function $\frac{1}{2}[\psi(z/2) - \psi(3/2 - z/2)] - \log 2$, where $\psi(z) = \Gamma'(z)/\Gamma(z)$. This function has simple poles at $z = -2k$, $k = 0, 1, 2, \ldots$, with the same residues as has $L(z)$. Thus the difference

$$R(z) = L(z) - \tfrac{1}{2}[\psi(z/2) - \psi(3/2 - z/2)] + \log 2 \tag{4.11}$$

has removable singularities at $z = -2k$, $k = 0, 1, 2, \ldots$. Using the representation (A 11) for $L(z)$, and some standard properties of the ψ-function (Abramowitz & Stegun 1965, formulae 6.3.7, 6.3.8), we deduce that $R(z)$ can be expressed as

$$R(z) = -\frac{1}{z-1} + \psi(1/2 - z/2) + E + 2\log 2 + P(z), \tag{4.12}$$

where $P(z)$ is given by (A 12). The Mellin transform (4.4) is now rewritten as

$$\mathscr{M}\{W(p)\} = \frac{\Gamma(z/2)\Gamma(3/2 - z/2)}{2\,\Gamma(3/2)}\{\tfrac{1}{2}[\psi(z/2) - \psi(3/2 - z/2)] - \log 2\} \\ + \frac{\Gamma(z/2)\Gamma(3/2 - z/2)}{2\,\Gamma(3/2)} R(z), \quad 1 < \mathrm{Re}(z) < 3. \tag{4.13}$$

Note that the first term on the right is the Mellin transform $\mathscr{M}\{(p^2+1)^{-3/2}\log(p/2)\}$, as can be seen by differentiating (4.3) with respect to z. Thus we have, by inversion of (4.13),

$$W(p) = (p^2 + 1)^{-3/2}\log(p/2) \\ + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z/2)\Gamma(3/2 - z/2)}{2\,\Gamma(3/2)} R(z)\, p^{-z}\mathrm{d}z, \quad 1 < c < 3. \tag{4.14}$$

The integrand in (4.14) is analytic to the left of the contour, except for a simple pole at $z = 1$, due to $R(z)$, and simple poles at $z = -2k$, $k = 0, 1, 2, \ldots$, due to $\Gamma(z/2)$. Accordingly, $W(p)$ can be represented by the residue series

$$W(p) \sim (p^2 + 1)^{-3/2}\log(p/2) \\ + \left[\operatorname*{Res}_{z=1} + \sum_{k=0}^{\infty} \operatorname*{Res}_{z=-2k}\right] \frac{\Gamma(z/2)\Gamma(3/2 - z/2)}{2\,\Gamma(3/2)} R(z)\, p^{-z} \quad (p \downarrow 0). \tag{4.15}$$

By use of (4.10) the residue at $z = 1$ is found to be

$$\frac{\Gamma(1/2)\Gamma(1)\,p^{-1}}{2\,\Gamma(3/2)} \operatorname*{Res}_{z=1} L(z) = p^{-1}. \tag{4.16}$$

The residue at $z = 0$ is given by

$$\frac{\Gamma(3/2)}{2\,\Gamma(3/2)} R(0) \operatorname*{Res}_{z=0} \Gamma(z/2) = R(0). \tag{4.17}$$

To find $R(0)$, we note that

$$P(0) = \sum_{n=1}^{\infty} \left[\frac{\Gamma(n+1/2)}{\Gamma(n+1/2)}\frac{1}{n} - \frac{1}{n}\right] = 0,$$

hence

$$R(0) = 1 + \psi(1/2) + E + 2\log 2 + P(0) = 1. \tag{4.18}$$

Next, the residue at $z = -2$ is given by

$$\frac{\Gamma(5/2)}{2\,\Gamma(3/2)}\, R(-2)\, p^2 \operatorname*{Res}_{z=-2}\, \Gamma(z/2) = -\tfrac{3}{2} R(-2)\, p^2. \tag{4.19}$$

The required value of $R(-2)$ is found from

$$P(-2) = \sum_{n=1}^{\infty}\left[\frac{\Gamma(n+3/2)}{\Gamma(n-1/2)}\,\frac{1}{n^3} - \frac{1}{n}\right] = \sum_{n=1}^{\infty}\left[\frac{n^2-1/4}{n^3} - \frac{1}{n}\right] = -\tfrac{1}{4}\zeta(3), \tag{4.20}$$

where $\zeta(\,\cdot\,)$ denotes the Riemann zeta-function, so that

$$R(-2) = \tfrac{1}{3} + \psi\left(\tfrac{3}{2}\right) + E + 2\log 2 + P(-2) = \tfrac{7}{3} - \tfrac{1}{4}\zeta(3). \tag{4.21}$$

By inserting these results into (4.15) we obtain the small-p expansion

$$W(p) = p^{-1} + (p^2+1)^{-3/2}\log(p/2) + 1 + \left[-\tfrac{7}{2} + \tfrac{3}{8}\zeta(3)\right] p^2 + O(p^4)\ (p \downarrow 0). \tag{4.22}$$

By substitution of (4.22) into (3.5), followed by a re-expansion for small p, we find

$$C_{MS} = -\tfrac{1}{4} + p^{-1} + \log p + 1 - \tfrac{1}{2}p + \left[\tfrac{3}{8}\zeta(3) - \tfrac{1}{2}\right] p^2 - \tfrac{5}{8}p^3 + O(p^4) \quad (p \downarrow 0) \tag{4.23}$$

which agrees with Kuibin & Okulov (1998, formula (25)), except that the term $-\tfrac{5}{8}p^3$ is missing in their result.

For the sake of completeness, and for use in discussing the numerical evaluation of $W(p)$, we note that generally in (4.15) the residue at $z = -2k$, $k = 0, 1, 2, \ldots$, is given by

$$\frac{\Gamma(k+3/2)}{2\,\Gamma(3/2)}\, R(-2k)\, p^{2k} \operatorname*{Res}_{z=-2k}\, \Gamma(z/2) = (-1)^k \frac{(3/2)_k}{k!}\, R(-2k)\, p^{2k}, \tag{4.24}$$

in which, by (4.12),

$$R(-2k) = \frac{1}{2k+1} + \sum_{m=1}^{k} \frac{1}{m-\frac{1}{2}} + P(-2k). \tag{4.25}$$

Thus the complete asymptotic expansion of $W(p)$ for small p reads

$$\begin{aligned} W(p) &\sim p^{-1} + (p^2+1)^{-3/2}\log(p/2) \\ &\quad + \sum_{k=0}^{\infty} (-1)^k \frac{(3/2)_k}{k!}\left[\frac{1}{2k+1} + \sum_{m=1}^{k}\frac{1}{m-\frac{1}{2}} + P(-2k)\right] p^{2k}\ (p \downarrow 0). \end{aligned} \tag{4.26}$$

The value of $P(-2k)$ is easily expressible as a linear combination of values $\zeta(2m+1)$, $m = 1, 2, \ldots, k$. For example, by setting $z = -4$ and $z = -6$ in (A 12), it is found that

$$P(-4) = \sum_{n=1}^{\infty}\left[\frac{(n^2-1/4)(n^2-9/4)}{n^5} - \frac{1}{n}\right] = -\tfrac{5}{2}\zeta(3) + \tfrac{9}{16}\zeta(5), \tag{4.27}$$

$$\begin{aligned} P(-6) &= \sum_{n=1}^{\infty}\left[\frac{(n^2-1/4)(n^2-9/4)(n^2-25/4)}{n^7} - \frac{1}{n}\right] \\ &= -\frac{35}{4}\zeta(3) + \frac{259}{16}\zeta(5) - \frac{225}{64}\zeta(7). \end{aligned} \tag{4.28}$$

The corresponding terms of the series in (4.26) are given by

$$\left[\frac{43}{8} - \frac{75}{16}\zeta(3) + \frac{135}{128}\zeta(5)\right] p^4 - \left[\frac{337}{48} - \frac{1225}{64}\zeta(3) + \frac{9065}{256}\zeta(5) - \frac{7875}{1024}\zeta(7)\right] p^6$$
$$= 0.834\, p^4 + 12.976\, p^6. \qquad (4.29)$$

Addition of these terms to (4.22) yields a small-p expansion that is correct up to order $O(p^8)$, as $p \downarrow 0$. For large values of k, the $n = 1$ term dominates the series for $P(-2k)$ in (A 12), so that

$$P(-2k) \sim \frac{\Gamma(3/2+k)}{\Gamma(3/2-k)} = (-1)^{k-1}\frac{k+1/2}{k-1/2}\,[(1/2)_k]^2 \quad (k \to \infty). \qquad (4.30)$$

It follows that the series in (4.26) is divergent for all $p > 0$, and is only an asymptotic expansion.

The numerical evaluation of $W(p)$ is made difficult, at small pitch, by the rapid change in the integrand as the vortex 'returns' every revolution to the immediate vicinity of the point at which the velocity is required. This, incidentally, is the reason for the large remainder term at small pitch. Whether the magnitude of this term is related to the torsion of the helical vortex, as implied by the title of Ricca's (1994) paper, is debatable, especially since C_{MS}, say, increases as p^{-1} as the torsion goes to zero.

We considered and tested a number of methods for numerically evaluating $W(p)$ given by (4.1), all of which proved difficult to implement accurately at small p, where t has to be very large for p^2t^2 to dominate the integrand. A change of variable to $u = 1/t$ leads to problems associated with the behaviour of $\sin(1/u)$ as $u \downarrow 0$. The asymptotic remainder method of Wood & Meyer (1991) for the n identical but displaced vortices from an n-bladed turbine or propeller, which relies on significant cancellation between the n integrands, was found to be inappropriate for the infinite integral for one vortex. The method finally chosen was based on rewriting (4.1) as

$$W(p) = A_0 + \sum_{k=1}^{N-1} A_k + W_N(p), \qquad (4.31)$$

where

$$A_0 = \int_0^{\pi} \left\{ \frac{\sin^2 t}{(p^2t^2 + \sin^2 t)^{3/2}} - \frac{1}{(p^2+1)^{3/2}}\,\frac{1}{t} \right\} \mathrm{d}t + \frac{\log(2\pi)}{(p^2+1)^{3/2}}, \qquad (4.32)$$

$$A_k = \int_0^{\pi} \frac{\sin^2 t}{[p^2(t+k\pi)^2 + \sin^2 t]^{3/2}}\,\mathrm{d}t, \qquad (4.33)$$

and

$$W_N(p) = \int_{N\pi}^{\infty} \frac{\sin^2 t}{(p^2t^2 + \sin^2 t)^{3/2}}\,\mathrm{d}t. \qquad (4.34)$$

Here, N is an integer such that $N\pi p > 1$. The integrals in (4.32) and (4.33) were evaluated using adaptive quadrature, as explained below. The integral $W_N(p)$ was approximated analytically in a manner that yields explicit error estimates and a suitable lower bound on N. The approximation is based on expanding the integrand in (4.34) in powers of p^{-1}, followed by a term-by-term integration. Retaining only the first two terms, we find

$$W_N(p) = B_0 p^{-3} - \tfrac{3}{2} B_1 p^{-5} + \sum_{m=2}^{\infty} (-1)^m \frac{(3/2)_m}{m!} \frac{B_m}{p^{2m+3}}, \tag{4.35}$$

where

$$B_m = \int_{N\pi}^{\infty} \frac{(\sin t)^{2m+2}}{t^{2m+3}}\, \mathrm{d}t. \tag{4.36}$$

Clearly

$$0 \leqslant B_m \leqslant \frac{(N\pi)^{-2m-2}}{2m+2},$$

which leads to an estimate for the final series in (4.35) of

$$\begin{aligned}\left| \sum_{m=2}^{\infty} (-1)^m \frac{(3/2)_m}{m!} \frac{B_m}{p^{2m+3}} \right| &\leqslant \frac{1}{2p} \sum_{m=2}^{\infty} \frac{(3/2)_m}{(m+1)!} (N\pi p)^{-2m-2} \leqslant \frac{5}{16} p^{-1} \sum_{m=2}^{\infty} (N\pi p)^{-2m-2} \\ &\leqslant \begin{cases} \dfrac{5}{16} p^{-1} \dfrac{(N\pi p)^{-6}}{1-(N\pi p)^{-2}}, & 0 < p \leqslant 1, \\ \dfrac{5}{16} p^{-7} \dfrac{(N\pi)^{-6}}{1-(N\pi)^{-2}}, & p \geqslant 1. \end{cases}\end{aligned} \tag{4.37}$$

Next, we evaluate B_0 and B_1 through repeated integration by parts. For B_0 we deduce

$$B_0 = \tfrac{1}{4}(N\pi)^{-2} - \tfrac{3}{8}(N\pi)^{-4} + \tfrac{15}{8}(N\pi)^{-6} + \varepsilon_0, \tag{4.38}$$

where

$$\varepsilon_0 = -\frac{315}{8} \int_{N\pi}^{\infty} \frac{\sin(2t)}{t^8}\, \mathrm{d}t$$

can be estimated by

$$|\varepsilon_0 p^{-3}| \leqslant \frac{315}{8p^3} \int_{N\pi}^{\infty} \frac{\mathrm{d}t}{t^8} = \frac{45}{8} (N\pi)^{-7} p^{-3}. \tag{4.39}$$

Similarly, B_1 is reduced to

$$B_1 = \frac{3}{32}(N\pi)^{-4} - \frac{75}{128}(N\pi)^{-6} + \varepsilon_1 \tag{4.40}$$

with the estimate

$$\left| \tfrac{3}{2} \varepsilon_1 p^{-5} \right| \leqslant \frac{1485}{512} (N\pi)^{-7} p^{-5}. \tag{4.41}$$

In evaluating $W(p)$, we aimed to achieve an accuracy of six significant digits. To that end, we select N as the smallest integer such that $N\pi \geqslant 10$ and $N\pi p \geqslant 10$. Then the estimates (4.37), (4.39) and (4.41) yield

$$\left| \sum_{m=2}^{\infty} (-1)^m \frac{(3/2)_m}{m!} \frac{B_m}{p^{2m+3}} \right| \leqslant \begin{cases} 0.32 p^{-1} 10^{-6}, & 0 < p \leqslant 1 \\ 0.32 p^{-7} 10^{-6}, & p \geqslant 1, \end{cases}$$

$$|\varepsilon_0 p^{-3}| \leqslant 0.57 \min\{1, p^{-3}\} 10^{-6}, \qquad \left| \frac{3}{2} \varepsilon_1 p^{-5} \right| \leqslant 0.3 \min\{1, p^{-5}\} 10^{-6}.$$

Since $W(p) \sim p^{-1}$ as $p \downarrow 0$, and $W(p) \sim (-E + 3/2)p^{-3}$ as $p \to \infty$, the desired accuracy is easily obtainable for the approximation to the integral $W_N(p)$.

Pitch	(*a*)	(*b*)	(*c*)	(*d*)
0.01	0.957022×10^{2}	0.957022×10^{2}	0.957022×10^{2}	
0.05	0.173173×10^{2}	0.173173×10^{2}	0.173173×10^{2}	
0.1	0.801822×10^{1}	0.801816×10^{1}	0.801824×10^{1}	
0.2	0.370710×10^{1}	0.370700×10^{1}	0.370834×10^{1}	
0.3	0.239240×10^{1}	0.239183×10^{1}	0.239858×10^{1}	
0.4	0.174543×10^{1}	0.172391×10^{1}	0.174526×10^{1}	
0.5	0.134138×10^{1}			
0.6	0.105695×10^{1}			
0.7	0.844909			
0.8	0.682350			
0.9	0.555822			
1.0	0.456367			0.499022×10^{-1}
2.0	0.928365×10^{-1}			0.880705×10^{-1}
3.0	0.308916×10^{-1}			0.305851×10^{-1}
4.0	0.136084×10^{-1}			0.135661×10^{-1}
5.0	0.711198×10^{-2}			0.710295×10^{-2}
6.0	0.416244×10^{-2}			0.415990×10^{-2}
7.0	0.263927×10^{-2}			0.263840×10^{-2}
8.0	0.177602×10^{-2}			0.177567×10^{-2}
9.0	0.125119×10^{-2}			0.125104×10^{-2}
10.0	0.914128×10^{-3}			0.914056×10^{-3}

TABLE 1. Numerical evaluation of $W(p)$ compared to the asymptotic expansions. (*a*) $W(p)$ determined by numerical quadrature; (*b*) $W(p)$ obtained from the small-p expansion (4.22); (*c*) $W(p)$ obtained from the small-p expansion (4.22) plus the terms (4.29); (*d*) $W(p)$ obtained from the large-p expansion (4.9).

The remaining integrals, in (4.32) and (4.33), were evaluated by adaptive quadrature, based simply on repeated bisection of the interval of integration, $[0, \pi]$, until the desired accuracy was achieved. Specifically, we used the Gauss–Kronrod rule in the form of routine GL15T described by Kahaner, Moler & Nash (1989). For the smallest pitch considered, $p = 0.01$, we obtained the following data: $N = 319$, $B_0 p^{-3} = 0.24892$, $-3/2B_1 p^{-5} = -1.394 \times 10^{-3}$; 780 evaluations of the integrands were required, of which 405 were used to determine $W(p)$; the routine's absolute error estimate was 2.7×10^{-7}. The program was checked against the Mathematica^R procedure NIntegrate which also uses the Gauss–Kronrod rule. It gave identical results to those shown in table 1. No attempt was made to optimize the subdivision of the interval $[0, \pi]$, or to explore ways of approximating the sum in (4.31) for large values of k. Nevertheless, we believe that all the tabulated values of $W(p)$ are accurate to the six significant digits given.

The numerical results for $W(p)$ have been collected in table 1. Column (*a*) comprises the values of $W(p)$ determined by the numerical quadrature procedure described above. The values of $W(p)$ in columns (*b*) and (*c*) are based on the small-p expansions (4.22), and (4.22) supplemented with the terms (4.29); these expansions are correct up to $O(p^4)$ and $O(p^8)$, respectively, as $p \downarrow 0$. The complete agreement between columns (*a*), (*b*) and (*c*), for $p = 0.01$ and $p = 0.05$, supports our belief that all the tabulated values are accurate to the six significant digits given. The final column (*d*) contains the values of $W(p)$ based on the large-p expansion (4.9). A comparison between columns (*a*) and (*b*) shows that the small-p expansion (4.22) is accurate to within 1% for $p \lesssim 0.4$, which covers virtually the whole range of interest for wind turbines. Because

$W(p)$ is the dominant term in C_{MS} or C_H at small pitch, this limit on the accuracy of $W(p)$ also applies to the full remainder term. For example, $C_{MS} = 2.00798$ when $p = 0.4$, accurate to six digits, as compared to $C_{MS} = 1.98109$ based on the small-p expansion of $W(p)$. Similarly, the large-p expansion (4.9) is accurate to within 10% for $p \gtrsim 1$ and within 1% for $p \gtrsim 5$.

5. Summary and conclusions

In this paper we have derived expressions for the remainder term appearing in the binormal velocity of an infinite helical vortex. The remainder was derived first from Hardin's (1982) solution for the inviscid flow around a line vortex of zero thickness. The primary result is the expression (2.10) for C_H. Then we derived in §3 the remainder from the Moore & Saffman (1972) procedure of calculating the binormal velocity by subtracting the curvature singularity resulting from a specific vortex structure: a circular core over which the vorticity is distributed uniformly. The main result is the expression (3.5) for C_{MS}, from which it follows immediately that C_{MS} and C_H differ by $1/4$ for all values of the vortex pitch. This generalizes the findings of Kuibin & Okulov (1998), who obtained this difference asymptotically at small and large pitch. It was pointed out that the difference between the remainders depends on the vortex structure, but is always independent of the pitch.

The analysis of Kuibin & Okulov (1998) began by replacing the Bessel functions in the Kapteyn series (2.2) and (2.12) by asymptotic expansions for small and large pitch. In §4 we describe a more consistent, but also more complex, analysis beginning with the only non-closed-form constituent of C_H and C_{MS}, namely, $W(p)$. A Mellin-transform method was used to obtain complete residue series for $W(p)$ at both large and small pitch. The asymptotic expansion for large p leads to binormal velocities consistent with the well-known Kelvin limit. The asymptotic expansion for small p reveals a minor correction to the expansion of Kuibin & Okulov (1998). One particular practical use for the small-p expansion is that $W(p)$ is closely related to the Biot–Savart integrals for the induced velocities on rotating blades in boundary-integral or panel methods used to predict the performance of wind turbines, propellers and rotors. The asymptotic analysis could be used to check the accuracy of quadrature schemes for the Biot–Savart integrals. Here, the calculation of the self-induced velocity at the smallest pitch is likely to be the most challenging problem.

The authors' names appear alphabetically. The mathematical analysis was done by J. B.; then, D. H. W. performed the numerical quadrature in §4 and wrote most of the paper. The collaboration arose from the problem submitted to *SIAM Review* by Wood & Guang (1997), and its solution by Boersma & Yakubovich (1998). We thank the editors of the Problem Section, Professors Cecil Rousseau and Otto Ruehr, for bringing us together in the final issue of the Problem Section. We also record our thanks to Professor Ruehr and Dr J. K. M. Jansen for their advice on, and checking of, the numerical evaluation of $W(p)$. D. H. W.'s work is funded by the Australian Research Council.

Appendix. Analyticity of $L(z)$

The function $L(z)$ as introduced in (4.5), is analytic for $\mathrm{Re}(z) > 1$. In this Appendix we show that $L(z)$ can be analytically continued as a meromorphic function with simple poles at $z = 1$ and $z = -2k$, $k = 0, 1, 2, \ldots$. We then determine the residues of $L(z)$ which are required for the small-p expansion (4.15) of $W(p)$. Finally, we evaluate $L(3)$ and $L(5)$ which are used in (4.8) to obtain the large-p expansion (4.9) of $W(p)$.

We start by rewriting $L(z)$ from (4.5) as

$$L(z) = \int_0^{\pi} \left[(\sin t)^{z-1} \sum_{k=0}^{\infty} \frac{1}{(t+k\pi)^z} - \frac{H(1/2-t)}{t} \right] \mathrm{d}t$$
$$= \int_0^{\pi} \left[(\sin t)^{z-1} \pi^{-z} \zeta\left(z, \frac{t}{\pi}\right) - \frac{H(1/2-t)}{t} \right] \mathrm{d}t, \qquad (\mathrm{A}\,1)$$

where ζ with two arguments denotes the generalized zeta-function, extensively discussed in Erdélyi *et al.* (1953, § 1.10). As a function of z, $\zeta(z, t/\pi)$ is analytic in the whole z-plane, except for a simple pole at $z = 1$ with

$$\operatorname*{Res}_{z=1} \zeta\left(z, \frac{t}{\pi}\right) = 1.$$

Furthermore

$$\pi^{-z}\zeta\left(z, \frac{t}{\pi}\right) \sim t^{-z} \quad \text{as} \quad t \downarrow 0,$$

so that the integral in (A 1) is convergent for $\mathrm{Re}(z) > 0$, $z \neq 1$. Consequently, $L(z)$ is analytic for $\mathrm{Re}(z) > 0$, except for a simple pole at $z = 1$ with

$$\operatorname*{Res}_{z=1} L(z) = \int_0^{\pi} \pi^{-1} \mathrm{d}t = 1. \qquad (\mathrm{A}\,2)$$

To avoid any difficulties with divergent integrals, we modify $L(z)$ into $L(z, \delta)$, defined by

$$L(z,\delta) = \int_0^{\pi} \left[(\sin t)^{z-1+\delta} \pi^{-z} \zeta\left(z, \frac{t}{\pi}\right) - \frac{H(1/2-t)}{t^{1-\delta}} \right] \mathrm{d}t, \quad \delta > 0 \qquad (\mathrm{A}\,3)$$

so that

$$\lim_{\delta \downarrow 0} L(z,\delta) = L(z). \qquad (\mathrm{A}\,4)$$

The second part of the integral (A 3) is readily evaluated as $-2^{-\delta}/\delta$. In the first part we substitute

$$\pi^{-z}\zeta\left(z, \frac{t}{\pi}\right) = \sum_{k=0}^{\infty} \frac{1}{(t+k\pi)^z} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(z)} \int_0^{\infty} s^{z-1} \mathrm{e}^{-(t+k\pi)s}\, \mathrm{d}s$$
$$= \frac{1}{\Gamma(z)} \int_0^{\infty} s^{z-1} \frac{\mathrm{e}^{-ts}}{1-\mathrm{e}^{-\pi s}}\, \mathrm{d}s, \qquad (\mathrm{A}\,5)$$

valid for $\mathrm{Re}(z) > 1$. Next, we interchange the order of integration and employ the auxiliary integral (Erdélyi *et al.* 1953, formula 1.5(29))

$$\int_0^{\pi} (\sin t)^{z-1+\delta} \mathrm{e}^{-ts} \mathrm{d}t$$
$$= \frac{\pi}{2^{z-1+\delta}} \frac{\Gamma(z+\delta)\,\mathrm{e}^{-\pi s/2}}{\Gamma(1/2+z/2+\delta/2+\mathrm{i}s/2)\Gamma(1/2+z/2+\delta/2-\mathrm{i}s/2)}. \qquad (\mathrm{A}\,6)$$

As a result we find

$$L(z,\delta)=-\frac{2^{-\delta}}{\delta}+\frac{\Gamma(z+\delta)}{\Gamma(z)}\frac{\pi}{2^{z-1+\delta}} \times\int_0^\infty \frac{s^{z-1}}{\Gamma(1/2+z/2+\delta/2+\mathrm{i}s/2)\Gamma(1/2+z/2+\delta/2-\mathrm{i}s/2)}\frac{\mathrm{e}^{-\pi s/2}}{1-\mathrm{e}^{-\pi s}}\,\mathrm{d}s. \quad \text{(A 7)}$$

Denote the integrand in (A 7) by $F(s)$, say, and observe that $F(s\mathrm{e}^{\pi\mathrm{i}})=\mathrm{e}^{\pi\mathrm{i}z}F(s)$, $s>0$. Then the integral can be extended to an integral along the whole real axis of the s-plane, to be evaluated by residue calculus:

$$\int_0^\infty F(s)\,\mathrm{d}s=\frac{1}{1+\mathrm{e}^{\pi\mathrm{i}z}}\int_{-\infty}^\infty F(s)\,\mathrm{d}s=\frac{\mathrm{e}^{-\pi\mathrm{i}z/2}}{2\cos(\pi z/2)}2\pi\mathrm{i}\sum_{n=1}^\infty \operatorname*{Res}_{s=2\mathrm{i}n}F(s). \quad \text{(A 8)}$$

Here, the integration path has been closed in the upper half-plane Im $(s)>0$, where $F(s)$ has simple poles at $s=2\mathrm{i}n$, $n=1,2,3,\ldots$. Finding the residues is straightforward and we are led to the representation

$$L(z,\delta)=-\frac{2^{-\delta}}{\delta}+\frac{\Gamma(z+\delta)}{\Gamma(z)}\frac{\cos[\pi(z+\delta)/2]}{\cos(\pi z/2)}2^{-\delta} \times\sum_{n=1}^\infty\frac{\Gamma(1/2-z/2-\delta/2+n)}{\Gamma(1/2+z/2+\delta/2+n)}\frac{1}{n^{1-z}}; \quad \text{(A 9)}$$

here, we used the reflection formula $\Gamma(1/2+v)\Gamma(1/2-v)=\pi/\cos(\pi v)$, cf. Abramowitz & Stegun (1965, formula 6.1.17). Since $\Gamma(n+a)/\Gamma(n+b)\sim n^{a-b}$ as $n\to\infty$, (Abramowitz & Stegun 1965, formula 6.1.47), the series in (A 9) compares to $\sum_{n=1}^\infty n^{-1-\delta}$ and is, therefore, convergent when $\delta>0$. Now we rewrite (A 9) as

$$L(z,\delta)=-\frac{2^{-\delta}}{\delta}+\frac{\Gamma(z+\delta)}{\Gamma(z)}\frac{\cos\,[\pi(z+\delta)/2]}{\cos\,(\pi z/2)}2^{-\delta} \times\left[\sum_{n=1}^\infty\left\{\frac{\Gamma(1/2-z/2-\delta/2+n)}{\Gamma(1/2+z/2+\delta/2+n)}\frac{1}{n^{1-z}}-\frac{1}{n^{1+\delta}}\right\}+\zeta(1+\delta)\right], \quad \text{(A 10)}$$

where $\zeta(\cdot)$ denotes the Riemann zeta-function. To obtain $L(z)$, we take the limit $\delta\downarrow 0$ in (A 10). In the evaluation of the limit we need the auxiliary results

$$\frac{\Gamma(z+\delta)}{\Gamma(z)}=1+\psi(z)\,\delta+O(\delta^2),$$
$$\frac{\cos\,[\pi(z+\delta)/2]}{\cos\,(\pi z/2)}=1-\frac{\pi}{2}\tan\,(\pi z/2)\,\delta+O(\delta^2),$$
$$\zeta(1+\delta)=\frac{1}{\delta}+E+O(\delta),$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$, and $E=0.5772\ldots$ is Euler's constant. The first two auxiliary results are straightforward, whereas the third result stems from Abramowitz & Stegun (1965, formula 23.2.5). Then the δ^{-1}-terms occurring in (A 10) cancel, as they should, and the limit is found to be

$$L(z)=\psi(z)-\frac{\pi}{2}\tan\,(\pi z/2)+P(z)+E, \quad \text{(A 11)}$$

where

$$P(z)=\sum_{n=1}^{\infty}\left\{\frac{\Gamma(1/2-z/2+n)}{\Gamma(1/2+z/2+n)}\,\frac{1}{n^{1-z}}-\frac{1}{n}\right\}. \tag{A 12}$$

The representation (A 11) for $L(z)$ is valid, first for $\mathrm{Re}(z)>1$, and next for all complex z by analytic continuation.

The representation (A 11) is well suited to find the singularities of $L(z)$. Consider first the series (A 12) for $P(z)$. Since (Abramowitz & Stegun 1965, formula 6.1.47)

$$\frac{\Gamma(1/2-z/2+n)}{\Gamma(1/2+z/2+n)}=n^{-z}[1+O(n^{-2})] \quad (n\to\infty)$$

the series (A 12) compares to $\sum_{n=1}^{\infty} n^{-3}$ and is convergent for all z. The terms in the series have simple poles at $z=2k+1$, $k=1,2,3,\ldots$, and

$$\begin{aligned}\operatorname*{Res}_{z=2k+1} P(z)&=\sum_{n=1}^{k}\frac{n^{2k}}{(k+n)!}\operatorname*{Res}_{z=2k+1}\Gamma(1/2-z/2+n)\\&=2(-1)^{k+1}\sum_{n=1}^{k}\frac{(-1)^n n^{2k}}{(k+n)!(k-n)!}=-1,\end{aligned} \tag{A 13}$$

where the finite sum can be obtained by a $(2k)$-fold differentiation of the Fourier series

$$(\sin t)^{2k}=\frac{1}{2^{2k}}\frac{(2k)!}{k!k!}+\frac{(2k)!}{2^{2k-1}}\sum_{n=1}^{k}\frac{(-1)^n}{(k+n)!(k-n)!}\cos(2nt)$$

at $t=0$. Additionally, $\psi(z)$ is analytic except for simple poles at $z=-k$, $k=0,1,2,\ldots$, with

$$\operatorname*{Res}_{z=-k}\psi(z)=-1;$$

$\tan(\pi z/2)$ is analytic except for simple poles at $z=2k+1$, $k=0,\pm1,\pm2,\ldots$, with

$$\operatorname*{Res}_{z=2k+1}\frac{\pi}{2}\tan(\pi z/2)=-1.$$

Using these results in (A 11), we find that $L(z)$ has removable singularities at $z=2k+1$, $k=\pm1,\pm2,\ldots$, since the residues of its constituents cancel. Consequently, $L(z)$ is analytic in the whole z-plane, except for simple poles at $z=1$ and $z=-2k$, $k=0,1,2,\ldots$, with

$$\operatorname*{Res}_{z=1} L(z)=-\operatorname*{Res}_{z=1}\frac{\pi}{2}\tan(\pi z/2)=1, \quad \operatorname*{Res}_{z=-2k} L(z)=\operatorname*{Res}_{z=-2k}\psi(z)=-1. \tag{A 14}$$

This completes the analysis of $L(z)$, so far as needed in the asymptotics of $W(p)$ for small pitch.

For large pitch we need the values of $L(3)$ and $L(5)$, to be substituted into (4.8). The representation (A 11) is less suitable for the evaluation. Therefore, we return to the original definition (4.5) of $L(z)$, which yields

$$L(2k+3)=\int_0^{\infty}\left[\frac{(\sin t)^{2k+2}}{t^{2k+3}}-\frac{H(1/2-t)}{t}\right]\mathrm{d}t. \tag{A 15}$$

The evaluation of (A 15) for $k = 0, 1$, proceeds through repeated integration by parts and use of the auxiliary integral (Abramowitz & Stegun 1965, formulae 5.2.2, 5.2.27)

$$\int_0^\infty \left[\frac{\cos\,(2at)}{t} - \frac{H(1/2-t)}{t}\right] \mathrm{d}t = \int_0^a \frac{\cos t - 1}{t} \mathrm{d}t + \int_a^\infty \frac{\cos t}{t} \mathrm{d}t = -E - \log a. \tag{A 16}$$

In this manner we find

$$\begin{aligned} L(3) &= \int_0^\infty \left[\frac{\sin^2 t}{t^3} - \frac{H(1/2-t)}{t}\right] \mathrm{d}t \\ &= -\left.\frac{\sin^2 t}{2t^2}\right|_0^\infty - \left.\frac{\mathrm{D}(\sin^2 t)}{2t}\right|_0^\infty + \int_0^\infty \left[\frac{\mathrm{D}^2(\sin^2 t)}{2t} - \frac{H(1/2-t)}{t}\right] \mathrm{d}t \\ &= \tfrac{1}{2} + 1 + \int_0^\infty \left[\frac{\cos\,(2t)}{t} - \frac{H(1/2-t)}{t}\right] \mathrm{d}t \\ &= -E + \tfrac{3}{2}, \end{aligned} \tag{A 17}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}t$. Similarly,

$$\begin{aligned} L(5) &= \int_0^\infty \left[\frac{\sin^4 t}{t^5} - \frac{H(1/2-t)}{t}\right] \mathrm{d}t \\ &= -\left.\frac{\sin^4 t}{4t^4}\right|_0^\infty - \left.\frac{\mathrm{D}(\sin^4 t)}{12t^3}\right|_0^\infty - \left.\frac{\mathrm{D}^2(\sin^4 t)}{24t^2}\right|_0^\infty - \left.\frac{\mathrm{D}^3(\sin^4 t)}{24t}\right|_0^\infty \\ &\quad + \int_0^\infty \left[\frac{1}{24t}\,\mathrm{D}^4\left(\tfrac{3}{8} - \tfrac{1}{2}\cos\,(2t) + \tfrac{1}{8}\cos\,(4t)\right) - \frac{H(1/2-t)}{t}\right] \mathrm{d}t \\ &= \tfrac{1}{4} + \tfrac{1}{3} + \tfrac{1}{2} + 1 - \tfrac{1}{3}\int_0^\infty \left[\frac{\cos\,(2t)}{t} - \frac{H(1/2-t)}{t}\right] \mathrm{d}t \\ &\quad + \frac{4}{3}\int_0^\infty \left[\frac{\cos\,(4t)}{t} - \frac{H(1/2-t)}{t}\right] \mathrm{d}t \\ &= -E + \tfrac{25}{12} - \tfrac{4}{3}\log 2. \end{aligned} \tag{A 18}$$

These results were used in obtaining (4.9).

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